A New Condition Implying The Existence of a Constant Mean Curvature Foliation

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Abstract

It is shown that if a non-flat spacetime (M, g) whose future c-boundary is a single point satisfies $R_{ab}V^aV^b \geq 0$ for all timelike vectors V^a, equality holding only if $R_{ab} = 0$, then sufficiently close to the future c-boundary the spacetime can be uniquely foliated by constant mean curvature compact hypersurfaces. The uniqueness proof uses a variational method developed by Brill and Flaherty to establish the uniqueness of maximal hypersurfaces.

In 1976 Dieter Brill and Frank Flaherty (1976) published an extremely important paper[1], "Isolated Maximal Hypersurfaces in Spacetime", establishing that maximal hypersurfaces are unique in closed universes with attractive gravity everywhere. That is, there is only one such hypersurface, if it exists at all. In an earlier paper, Brill had established that in three-torus universes, only suitably identified flat space possessed a maximal hypersurface, so the existence of a maximal hypersurface is not guaranteed. These results by Brill are important because maximal hypersurfaces are very convenient spacelike hypersurfaces upon which to impose initial data; on such hypersurfaces the constraint equations are enormously simplified. Furthermore, in asymptotically flat space, foliations of spacetime by maximal hypersurfaces often exist, and the simplifications of the constraint equations on such a foliation make it

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[1] I regard this paper of Dieter's as important because I've used its results in about ten of my own papers. But the relativity community also finds this paper important. According to the Science Citation Index, it has been cited 6 times in the period January 1989 through August of 1992, thus gathering about 6 per cent of Dieter's total citation count of 102 for this period. Most papers are never cited ten years after publication, so this citation frequency is quite impressive.

easy to numerically solve[2] the full four-dimensional vacuum Einstein equations for physically interesting situations.

Constant mean curvature foliations give similar simplifications, and such foliations often exist in closed universes. As Brill and Flaherty realized, their method can be generalized to show that if such a foliation exists, then it is unique. What I shall do in this paper is establish the existence of such a foliation near the final singularity in the case that the singularity is an "omega point". I shall conclude this paper with a discussion on the connection between Penrose's Weyl Curvature Hypothesis and the existence of a foliation of the entire spacetime by constant mean curvature hypersurfaces.

Let me begin with a

Definition. A spacetime (M,g) will be said to terminate in an *omega point* if its future c-boundary consists of a single point.

I have discussed elsewhere (Barrow & Tipler 1986, Tipler 1986, 1989, 1992) reasons for believing that the actual universe may terminate in an omega point. Misner's Mixmaster universe (Misner 1967) was specifically constructed to have a point c-boundary in the past, though it is not known if in fact there is a vacuum Bianchi type IX universe with such a c-boundary. Doroshkevich *et al* (1971) established (using different terminology) that if such a vacuum solution exists, it is of very small measure in the vacuum Bianchi type IX vacuum initial data. Vacuum solutions to the Einstein equations which terminate in an omega point are known (Löbell 1931; Hawking & Ellis 1973, p. 120 & p. 205; Budic & Sachs 1976), but they are all locally flat.

Let me give two examples of S^3 spatial topology Friedmann universes which terminate in omega points. Recall that the S^3 Friedmann metric is

$$ds^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)] \tag{1}$$

where $0 \le \chi \le \pi$, $0 \le \theta \le \pi$, and $0 \le \phi < 2\pi$. In the Friedmann universe, all null geodesics are radial, with comoving coordinates given by $ds^2 = 0 = -dt^2 + a^2(t)d\chi^2$, which upon integration yields Rindler's equation:

$$\chi_f - \chi_i = \pm\int_{t_i}^{t_f} \frac{dt}{a(t)} \tag{2}$$

[2]I leave this split infinitive in for Dieter's amusement. When I was his graduate student in the early 1970's, he was always finding them in my drafts of papers. (As a German, he naturally had a better command of English than a native American.) But just before I received my Ph.D. from him in 1976, he circulated a note announcing that, since he had just found a large number of split infinitives in the *Congressional Record*, he would henceforth regard split infinitives as officially correct American English.

The future c-boundary is a single point if and only if the integral in equation (2) diverges as proper time approaches its future limit $t_f = t_{max}$, since only in this case will the event horizons disappear: light rays circumnavigate the universe an infinity of times no matter how close to the c-boundary one is.

example 1: If $a(t) =$ constant, the metric (1) represents the Einstein static universe. Since the integral (2) diverges in this case because $t_{max} = +\infty$ and $t_{min} = -\infty$, both the future and past c-boundaries are single points.

example 2: Let $a(t) = \sin t$. Then the integral (2) is $\ln|\frac{\tan(t_f/2)}{\tan(t_i/2)}|$. There are s.p. curvature singularities (Hawking & Ellis 1973) at $t_f = \pi$ and at $t_i = 0$. At either of these limits, the integral (2) diverges, so this example has the same c-boundary structure as the Einstein static universe: both the future and past c-boundaries are each single points.

The Friedmann universe of example 2 does not satisfy the Einstein equations with any standard equation of state. However, this example is worth analysis for two reasons: first, because this $a(t)$ can be smoothly joined to a Friedmann universe which is matter and/or radiation dominated to the future of 10^{-15} seconds (before which $a(t) = \sin t$ may be appropriate, for who knows what the stress-energy tensor is like at extremely high densities), and second, because it nevertheless obeys all the usual energy conditions, thus showing that even in the case of the closed Friedmann universe, one need not violate the energy conditions to get the future and past c-boundaries to be single points. (This example is thus a counter-example to a conjecture of Budic and Sachs (1976), that to have a single point as its c-boundary, "... a cosmological model may have to 'coast into the [singularity] so slowly it almost bounces' corresponding to a 'near violation' of the timelike convergence condition. (Budic & Sachs 1976, p. 28)". But the metric of example 2 does not "nearly violate" the timelike convergence condition.

To see this, let us compute the stress-energy tensor for the metric of example 2. The mass density is

$$\mu \equiv T_{\hat{t}\hat{t}} \equiv \frac{1}{8\pi}G_{\hat{t}\hat{t}} = \frac{3}{8\pi}\left(\frac{a'^2+1}{a^2}\right) = \frac{3}{8\pi}\left(\frac{\cos^2 t+1}{\sin^2 t}\right) \geq \frac{3}{8\pi}$$

The principal pressure is

$$p \equiv T_{\hat{\chi}\hat{\chi}} \equiv \frac{1}{8\pi}G_{\hat{\chi}\hat{\chi}} = -\left(\frac{1}{8\pi}\right)\frac{2aa''+a'^2+1}{a^2} = \frac{1}{8\pi}\left(1-2\cot^2 t\right)$$

which is negative for $|\cot t| > \frac{1}{\sqrt{2}}$ — that is, near the singularities — and $p \to -\infty$ as $t \to 0$ or π. But we have

$$\mu + p = \frac{1}{8\pi}\left(\frac{4}{\sin^2 t}\right) > \frac{1}{2\pi} \quad , \quad \mu + 3p = \frac{6}{8\pi}$$

Since the weak energy condition requires (Hawking & Ellis 1973) $\mu \geq 0$ and $\mu + p \geq 0$, the weak energy condition is satisfied. Since the strong energy condition (here, also the timelike convergence condition) requires $\mu + p \geq 0$ and $\mu + 3p \geq 0$, the strong energy condition is satisfied. Furthermore, since both $\mu + p$ and $\mu + 3p$ are bounded well away from zero at *all* times, the timelike convergence condition is never "nearly violated". The dominant energy condition (Hawking & Ellis, 1973) requires $\mu \geq 0$ and $-\mu \leq p \leq +\mu$, so the dominant energy condition is satisfied. Finally it is easily checked that the generic condition is satisfied. The Ricci scalar is $R = 6(aa'' + a^2 + 1)/a^2 = 6\sin^{-2} t$, so the single c-boundary points are true s.p. curvature singularities at $t = 0$ and at $t = \pi$.

Note that examples 1 and 2 collectively suggest that if the closed Friedmann universe is not the Einstein static universe, then negative pressures are required in order for the c-boundary to be a single point. This can in fact be proven, but I shall omit the proof.

Budic and Sachs (1976) were motivated by Misner's model to prove some general theorems on spacetimes with either the future or the past c-boundaries being single points. Their theorems can be applied to either the past or the future c-boundary, though they stated their theorems in terms of a single point c-boundary in the past (since they were thinking of Misner's model). Similarly, though I shall state the theorems below in terms of an omega point — I shall be thinking of the final rather than the initial singularity — the theorems can be trivially modified to apply to the initial singularity.

Requiring that a spacetime end in an omega point imposes very powerful constraints on the spacetime. For example,

Theorem (Seifert 1971): a spacetime which terminates in an omega point and which satisfies the chronology condition has a compact Cauchy hypersurface.

This theorem was first stated by Seifert (1971), but unfortunately his proof is defective (In his Theorem 6.3, Seifert claims that the existence of an omega point in both the past and future directions is equivalent to the existence of a compact Cauchy surface). Budic and Sachs (1976) have stated that the existence of an omega point in a future and past distinguishing spacetime (Hawking & Ellis 1973) implies the existence of a compact Cauchy surface. It is easy to check that Seifert's Theorem holds if the spacetime is stably causal, so I've stated his Theorem with this causality condition.

As a converse to Seifert's Theorem, we have

Theorem 1: If the future c-boundary of a stably causal spacetime consists of an omega point, then for all points q sufficiently close to the future c-boundary, $\partial I^-(q)$ is also a Cauchy surface.

Proof: By Seifert's Theorem, the spacetime admits a compact Cauchy surface. Since the spacetime has a compact Cauchy surface, Geroch's Theorem (Proposition 6.6.8 of Hawking & Ellis 1973), page 212), all Cauchy surfaces in the spacetime have the same topology, and further, the spacetime can be foliated by compact diffeomorphic spacelike Cauchy surfaces. Let $S(t)$ represent such a foliation, where t increases in the future direction, and let $V^a(\vec{x},t)$ represent the timelike future-directed unit vector field which is everywhere normal to S(t). Let $\lambda(t)$ be any flow line of this vector field. I claim that there exists t_λ such that $\partial I^-(\lambda(t_\lambda))$ is a Cauchy surface. Suppose not. Then there would exist another flow line $\mu(t)$ of $V^a(\vec{x},t)$ which never intersects $\partial I^-(\lambda(t))$, for any t. But then the flow line $\mu(t)$ would define a different future c-boundary point than $\lambda(t)$, contrary to the fact that there is only one c-boundary point. Thus for each $\lambda(t)$ in $V^a(\vec{x},t)$, there is a time t_λ for which $\partial I^-(\lambda(t))$ is a Cauchy surface, for all $t > t_\lambda$. Since the leaves of the foliation $S(t)$ are compact, $sup[t_\lambda] \equiv t_C$ is achieved in the spacetime. Then $\partial I^-(q)$ will be a Cauchy surface provided q is any event to the future of $S(t_C)$; i.e., $q \in I^+(S(t_C)$. QED

We thus know that $\partial I^-(q)$ is a Cauchy surface for q sufficiently close to the omega point, so in principle, all information is available at q. This property allows us to show that a foliation of spacetime by constant mean curvature hypersurfaces exists, at least sufficiently near the omega point.

Theorem 2: If a non-flat stably causal spacetime (M,g) satisfies $R_{ab}V^aV^b \geq 0$ for all timelike vectors V^a, equality holding only if $R_{ab} = 0$, and (M,g) has an omega point, then there exists a point $p \in M$ such that through p there passes a $C^{2,\alpha}$ Cauchy surface S with constant mean curvature, and further, $I^+(S)$ can be uniquely foliated by $C^{2,\alpha}$ Cauchy surfaces with constant mean curvature.

That is, a spacetime which satisfies the timelike convergence condition and which ends in an omega point has sufficiently near the omega point a foliation by compact Cauchy surfaces with constant mean curvature. However, the entire spacetime might not have such a foliation; the foliation is guaranteed to exist only for that part of spacetime sufficiently close to the omega point. The meaning of "sufficiently close" is made precise in the proof of Theorem 1 above. (A $C^{2,\alpha}$ Cauchy surface (Bartnik 1984) is one which is C^2 with these second derivatives being Hölder continuous of order α.)

Proof: Bartnik (1988) has shown that if for any point p in (M,g), the set $M - I^+(p) \cup I^-(p)$ is compact, then there is a spacelike $C^{2,\alpha}$ constant mean curvature Cauchy surface through p. I shall need two Lemmas to combine with Bartnik's result:

Lemma 1: If the future c-boundary of a spacetime (M,g) which satisfies the chronology condition is an omega point, then the achronal boundary $\partial I^+(p)$ is a Cauchy surface for *any* point p in the spacetime.

Proof: Suppose not. Then there is a future- and past-endless timelike curve γ which never intersects $\partial I^+(p)$, which, since the chronology condition holds, is non-empty and is generated by null geodesic segments at least some of which intersect p. If (1) $\gamma \cap I^-(\partial I^+(p)) \neq \emptyset$, or (2) $\gamma \cap I^-(\partial I^+(p)) = \emptyset$ and $\gamma \cap I^+(\partial I^+(p)) = \emptyset$, then $I^-(\gamma)$ would not intersect $I^+(p)$, so $I^-(\gamma)$ defines a different c-boundary point than does a future-endless timelike curve which eventually enters $I^+(p)$. Thus there are at least two distinct c-boundary points, contradicting the hypothesis that there is just one future c-boundary point.

The other possibility, which we now eliminate, is $\gamma \cap I^+(p) \neq \emptyset$, but $\gamma \cap \partial I^+(p) = \emptyset$. Since $\gamma \cap I^+(p) \neq \emptyset$, there exists a timelike curve β_q from p to *some* point $q \in \gamma$. Consider the sequence of timelike curves β_{q_i} as the point q moves into the past along γ through a sequence of points q_i. This sequence defines a subsequence which converges to some causal curve $\hat{\beta}$ in $\overline{I^+(p)}$ (since $\overline{I^+(p)}$ is closed). However, $\hat{\beta}$ must be disconnected since if it were connected, $\gamma \cup \hat{\beta}$ would be a connected curve, contrary to the assumption that γ is past-endless. The connected subset of $\hat{\beta}$ — call it $\hat{\beta}_p$ — which ends in the point p is thus future-endless, and since $\overline{I^+(\gamma)} \cap I^-(\hat{\beta}_p) = \emptyset$, the causal curves γ and $\hat{\beta}_p$ define different TIPs, contrary to the assumption that there is just once TIP in (M, g). QED.

Lemma 2: If the future c-boundary of $M, g)$ is a single point and the chronology condition holds, then $\partial I^+(p)$ is non-empty and compact for every event p in the spacetime.

Proof: If the chronology condition holds, then $p \in \partial I^+(p)$. By the remarks on page 188 of Hawking and Ellis (1973), $\partial I^+(p)$ is generated by null geodesic segments which either have no endpoints or have endpoints at p. Thus all the null geodesics from p into the future are generators of $\partial I^+(p)$. If every null geodesic generator of $\partial I^+(p)$ from p leaves $\partial I^+(p)$ in the future, then $\partial I^+(p)$ is compact, since one can put on the collection of null geodesic generators of $\partial I^+(p)$ an affine parameterization such that the length of the segment of the null geodesic in $\partial I^+(p)$ from p varies continuously with the null direction into the future from p, and the collection of null directions at p is compact (actually, a 2-sphere). Thus the only way that part of $\partial I^+(p)$ for which $\partial I^+(p) \cap \{p\} \neq \emptyset$ could fail to be compact is for there to exist a null geodesic γ of $\partial I^+(p)$ which never leaves $\partial I^+(p)$.

But then $I^-(\gamma)$ would define a TIP which is distinct from a TIP generated by any future-inextendible timelike curve which crossed γ from $I^-(\gamma)$ into $I^+(p)$. But this would mean more than one TIP, contrary to assumption, so that part of $\partial I^+(p)$ for which $\partial I^+(p) \cap \{p\} \neq \emptyset$ is compact for all p, and also all null geodesic generators of $\partial I^+(p)$ from p must eventually leave $\partial I^+(p)$.

We now eliminate the possibility that $\partial I^+(p)$ has a null geodesic generator β which does not intersect p. Suppose it does, and let q be a point of β with normal neighborhood N. Then there is a timelike curve from p to any point in $N \cap I^+(p)$, (which is non-empty since $\beta \subset \partial I^+(p)$.) Consider a sequence of points q_i in $N \cap I^+(p)$ converging to q. This sequence defines a sequence of timelike curves β_i from p to q_i. If this sequence of timelike curves converged to a single connected causal curve, it would have to be a null geodesic with past endpoint at p, which is impossible by definition of β. Since locally (in any convex normal neighbourhood) the sequence converges, it must converge globally to at least two (possibly more) distinct disconnected causal curves, the one terminating at p being future-endless. This future-endless curve, call it $\hat{\beta}$, defines a TIP which is different from at least one TIP defined by some future-endless timelike curve in $I^+(\beta)$, since by construction $I^-(\hat{\beta}) \cap I^+(\beta) = \emptyset$. QED

To continue the proof of Theorem 2, recall that by Theorem 1 above, $\partial I^-(p)$ is a compact Cauchy surface for all points p sufficiently close to the omega point. Together these imply that $M - [I^+(p) \cup I^-(p)]$ is compact for p sufficiently close to the omega point. (The set $M - [I^+(p) \cup I^-(p)]$ is closed since both $I^+(p)$ and $I^-(p)$ are open). Also, for any foliation of (M, g) by spacelike hypersurfaces $S(t)$, there will times t_1 and t_0 with $t_1 > t_0$ such that $\partial I^+(p) \subset I^-(S(t_1))$ and $\partial I^-(p) \subset I^+(S(t_0))$. Hence, the closed set $M - [I^+(p) \cup I^-(p)]$ is contained in the compact set $M - [I^+(S(t_1)) \cup I^-S(t_0))] \approx S(t) \times [0, 1]$, for any fixed t, and so is compact.) Thus through every point sufficiently close to the omega point, there passes a spacelike $C^{2,\alpha}$ constant mean curvature Cauchy surface. Brill and Flaherty (1976), and Marsden and Tipler (1980) have modified a theorem by Brill and Flaherty (1976) to show that any constant mean curvature compact Cauchy surface on which the constant mean curvature $\chi^a{}_a$ is non-zero, is unique if the timelike convergence condition holds. Following Geroch (see Hawking and Ellis 1973, p. 274), Marsden and Tipler (1980) have shown that in all non-flat spacetimes with $R_{ab}V^aV^b \geq 0$, equality holding only if $R_{ab} = 0$, compact Cauchy surfaces with $\chi^a{}_a = 0$ are also unique. Hence, there exists a point p in M such that through p there passes a $C^{2,\alpha}$ Cauchy surface S with constant mean curvature, and further, $I^+(S)$ can be uniquely foliated by Cauchy surfaces with constant mean curvature. QED.

The non-flatness and $R_{ab}V^aV^b = 0$ only when $R_{ab} = 0$ assumptions were only needed for uniqueness of the maximal hypersurface (if in fact any exists). The existence of a constant mean curvature compact Cauchy surface foliation follows merely from the timelike convergence condition and the existence of the omega point. If both the future and past c-boundaries are single points — as in examples 1 and 2 — then the proof of Theorem 2 shows that the entire spacetime is foliated by constant mean curvature compact Cauchy surfaces.

Budic and Sachs (1976) have shown that if the total spacetime volume $\int \sqrt{-g}\, d^4x$ of an omega point spacetime is finite (as it would be in example 2, for instance), then there is another natural foliation $S_{BS}(t)$ of (M, g) by spacelike hypersurfaces, namely for a given t, the value of $\int_{I^+(p)} \sqrt{-g}\, d^4x$ is the same for each point $p \in S_{BS}(t)$. Budic and Sachs show that this foliation is C^1, and a modification of the proof of Theorem 2 shows that sufficiently close to the omega point, the hypersurfaces $S_{BS}(t)$ will be compact Cauchy surfaces. The question then arises, what is the relationship — if any — between these two natural spacelike foliations of (M, g)? In example 2, the two foliations are exactly the same, but in general this will not be the case. For instance, if (M, g) is the spacetime of example 2, then $M - J^-(p)$ for any point $p \in M$ is a spacetime with an omega point which can be foliated with constant mean curvature Cauchy surfaces only to the future of p, while $S_{BS}(t)$ foliate the entire spacetime (though with Cauchy surfaces only to the future of p).

Budic and Sachs (1976) show that $\overline{M}$, the spacetime with its c-boundary, is second-countable and metrizable, so some constraints are imposed on the initial singularity by the requirement that the final singularity is an omega point. I conjecture that if we require that the entire spacetime be foliated by constant mean curvature Cauchy surfaces which everywhere coincide with the $S_{BS}(t)$ hypersurfaces, then the spacetime must be spatially homogeneous.

Penrose's Weyl Curvature Hypothesis (Penrose 1979), namely that time is defined so that a physical spacetime's "initial" singularity is characterized by the vanishing of the Weyl curvature as one approaches the initial singularity (and the "final" singularity is characterized by the dominance of the Weyl curvature over the Ricci curvature) is another proposal to connect the initial and final singularities. Tod (1990) conjectured and Newman (1991) proved (at least for the $\gamma = \frac{4}{3}$ case) that if the Weyl curvature vanished at a singularity (which is "conformally compactifiable"), then the spacetime was necessarily Friedmann everywhere. Goode *et al* (1985, 1991, 1992) have restated the Weyl Curvature Hypothesis to mean that

$$\lim_{T \to 0^+} \frac{C_{abcd}C^{abcd}}{R_{ab}R^{ab}} = 0 \tag{3}$$

at an "initial" singularity. Goode *et al* (1992) have shown that many of the standard Cosmological Problems (flatness problem, horizon problem, etc.) can be solved if one imposes this modified Weyl Curvature Hypothesis. However, they do not propose strongly believable reasons *why* the Weyl Curvature Hypothesis should be true.

Perhaps by connecting these two approaches to connecting the initial and final singularies a strongly believeable reason can be found. Goode *et al* and Tod, in their definitions of "conformally comfactifiable" or "isotropic" singularity, require the existence, near the initial singularity, of a foliation of spacetime by spacelike hypersurfaces, but they do not require that the foliation be one of the "natural" ones discussed above.

However, suppose we require that globally, the Second Law of Thermodynamics must *always* hold: the total entropy of the universe at time t_i must always be greater than or equal to the total entropy at time t_j whenever $t_i \geq t_j$. Clearly, this inequality cannot hold globally for all foliations, since locally we can always decrease the entropy at the expense of an even greater entropy increase at another spatial position, and we can use this fact to construct a foliation of spacetime by spacelike hypersurfaces in which the above entropy inequality was violated, at least for a short time. But it conceivably *might* be true for one (or both) of the natural foliations described above — if the modified Penrose Weyl Curvature Hypothesis holds. If the entropy inequalities do not hold for *some* natural foliation, then we would be forced to admit that the Second Law of Thermodynamics simply does not always hold globally (or is inconsistent with general relativity), an admission we should be loath to make.

The modified Penrose Weyl Curvature Hypothesis would have to hold for two reasons. First, to ensure that the purely gravitation degrees of freedom — gravitational waves — when degraded into heat, do not by themselves violate the Second Law of Thermodynamics. Second, to ensure the global existence of both of the above foliations: I conjecture that if the initial singularity is "isotropic" in the sense of Goode *at al* and "conformally compactifiable" in the sense of Tod, then the foliation of constant mean curvature Cauchy surfaces and the Budic-Sachs foliation by Cauchy surfaces — which must exist near an omega point — can be extended globally to the entire spacetime.

If so, then the modified Penrose Weyl Curvature Hypothesis would be equivalent to requiring the global validity of the Second Law of Thermodynamics. Here would be a strongly believable reason for accepting the Weyl Curvature Hypothesis and its resolution of the Cosmological Problems! This modified Penrose Curvature Hypothesis also gives another reason for studying constant mean curvature foliations, a research topic to which Dieter Rolf Brill has contributed so much.

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REFERENCES

Barrow, J.D. & Tipler, F.J. (1986). *The Anthropic Cosmological Principle.* Oxford University Press: Oxford.

Bartnik, R. (1984). *Commun. Math. Phys.*, **94**, 155.

Bartnik, R. (1988). *Commun. Math. Phys.*, **117**, 615.

Brill, D.R. & Flaherty, F. (1976). *Comm. Math. Phys.*, **50**, 157.

Budic, R. & Sachs, R.K. (1976). *Gen. Rel. Grav.*, **7**, 21.

Doroshkevich, A.G. & Novikov, I.D. (1971). *Sov. Astron. AJ*, **14**, 763.

Doroshkevich, A.G., Lukash, V.N., & Novikov, I.D. (1971). *Sov. Phys. JETP*, **33**, 649.

Goode, S.W. (1991). *Class. Quantum Grav.*, **8**, L1.

Goode, S.W., Coley, A.A. & Wainwright, J. (1992). *Class. Quantum Grav.*, **9**, 445.

Goode, S.W. & Wainwright, J. (1985). *Class. Quantum Grav.*, **2**, 99.

S.W. Hawking, S.W. & Ellis, G.F.R. (1973). *The Large- Scale Structure of Space-Time.* (Cambridge University Press: Cambridge.

Lifshitz, E.M., Lifshitz, I.M. & Khalatnikov, I.M. (1971). *Sov. Phys. JETP*, **32**, 173.

Löbell, F. (1931). *Ber. Verhandl. Sächs. Akad. Wiss. Leipzig, Math. Phys. Kl.*, **83**, 167.

Marsden, J.E. & Tipler, F.J. (1980). *Phys. Rep.*, **66**, 109.

Misner, C.W. (1967). *Nature*, **214**, 40.

Newman, R.P.A.C. (1991). *Twistor Newsletter*, **33**, 11.

Penrose, R. (1979) in *General Relativity: An Einstein Centenary Survey*, ed. S.W. Hawking & W. Israel. Cambridge Univeristy Press: Cambridge.

Seifert, H.J. (1971). *Gen. Rel. Grav.*, **1**, 247.

Tipler, F.J. (1986). *Int. J. Theor. Phys.*, **25**, 617.

Tipler, F.J. (1989). *Zygon*, **24**, 217.

Tipler, F.J. (1992). *Phys. Lett. B*, **286**, 36.

Tod, K.P. (1990). *Class. Quantum Grav.*, **7**, L13.